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Study of Fractional Gaussian Integral

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Abstract: **Based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, this paper studies the fractional Gaussian integral. A new multiplication of fractional analytic functions plays an important role in this article. In fact, our result is the generalization of ordinary calculus result.**

Keyword: **Jumarie type of R-L fractional calculus, fractional Gaussian integral, new multiplication, fractional analytic functions.**

I. INTRODUCTION

In the first half of the 19th century, Abel, Liouville and Riemann correctly introduced fractional integral and derivative in the analysis. However, the use of generalized differential and integral operators became more familiar in the last decades of the 19th century because of the symbolic calculus of Heaviside and the work of mathematicians such as Hadamard, Hardy and Littlewood, Riesz, and Weyl. The application of fractional calculus in many fields such as physics, biology, economics, and engineering has aroused great interest [1-9].

However, the definition of fractional derivative is not unique. The commonly used definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [10-13]. Since Jumarie's modified R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie type of R-L fractional derivative, we evaluate the following α-fractional Gaussian integral:

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[E_{\alpha}\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\right], \tag{1}$$

where $0<\alpha\leq 1$, and $(-1)^{\alpha}$ exists. A new multiplication of fractional analytic functions plays an important role in this article. And our result is the generalization of the result in traditional calculus.

II. PRELIMINARIES

Firstly, the fractional derivative used in this paper and its properties are introduced below.

Definition 2.1 ([14])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{2}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I^{\alpha}_{x}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{3}$$

where $\Gamma(\ )$ is the gamma function.

Proposition 2.2 ([15])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_0D_x^\alpha\right)\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, \tag{4}$$

and

$$\left({}_0D_x^\alpha\right)[C] = 0. \tag{5}$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([16])**:** Suppose that x and a_k are real numbers for all k, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a, b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha}$, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic function.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([17])**:** If $0 < \alpha \leq 1$. Suppose that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}, \tag{6}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{7}$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} \otimes \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} \\ & = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right) x^{k\alpha}. \end{aligned} \tag{8}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes g_\alpha(x^\alpha) \\ & = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} \otimes \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k} \\ & = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\right) \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \end{aligned} \tag{9}$$

Definition 2.5 ([18])**:** Assume that $0 < \alpha \leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are α-fractional analytic functions,

$$f_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}, \tag{10}$$

$$g_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)} x^{k\alpha} = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{11}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha \circ g_\alpha)(x^\alpha) = f_\alpha\left(g_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{a_k}{k!} \left(g_\alpha(x^\alpha)\right)^{\otimes k}, \tag{12}$$

and

$$(g_\alpha \circ f_\alpha)(x^\alpha) = g_\alpha\left(f_\alpha(x^\alpha)\right) = \sum_{k=0}^{\infty} \frac{b_k}{k!} \left(f_\alpha(x^\alpha)\right)^{\otimes k}. \tag{13}$$

Definition 2.6 ([19])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(k\alpha+1)} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes k}. \tag{14}$$

III. MAIN RESULT

In this section, we evaluate the fractional Gaussian integral. At first, two lemmas are needed.

Lemma 3.1: *Let* $0<\alpha\leq 1$, *and* $f_\alpha(x^\alpha)$, $g_\alpha(y^\alpha)$ *be* α*-fractional analytic functions at* 0, *then*

$$\left({}_0I^{\alpha}_{+\infty}\right)\left({}_0I^{\alpha}_{+\infty}\right)[f_\alpha(x^\alpha)\otimes g_\alpha(y^\alpha)]=\left({}_0I^{\alpha}_{+\infty}\right)[f_\alpha(x^\alpha)]\cdot\left({}_0I^{\alpha}_{+\infty}\right)[g_\alpha(y^\alpha)]. \tag{15}$$

Proof

$$\left({}_0I^{\alpha}_{+\infty}\right)\left({}_0I^{\alpha}_{+\infty}\right)[f_\alpha(x^\alpha)\otimes g_\alpha(y^\alpha)]$$
$$=\lim_{\substack{x\to+\infty\\y\to+\infty}}\left({}_0I^{\alpha}_{y}\right)\left({}_0I^{\alpha}_{x}\right)[[f_\alpha(x^\alpha)\otimes g_\alpha(y^\alpha)]]$$
$$=\lim_{\substack{x\to+\infty\\y\to+\infty}}\left({}_0I^{\alpha}_{x}\right)[f_\alpha(x^\alpha)]\otimes\left({}_0I^{\alpha}_{y}\right)[g_\alpha(y^\alpha)]$$
$$=\lim_{x\to+\infty}\left({}_0I^{\alpha}_{x}\right)[f_\alpha(x^\alpha)]\otimes\lim_{y\to+\infty}\left({}_0I^{\alpha}_{y}\right)[g_\alpha(y^\alpha)]$$
$$=\left({}_0I^{\alpha}_{+\infty}\right)[f_\alpha(x^\alpha)]\cdot\left({}_0I^{\alpha}_{+\infty}\right)[g_\alpha(y^\alpha)].$$

Q.e.d.

Lemma 3.2: *If* $0<\alpha\leq 1$, t *is a real number, and* $f_\alpha(x^\alpha)$ *is a* α*-fractional analytic function at* $x=0$, *then*

$$\left({}_0I^{\alpha}_{x}\right)\left[f_\alpha(tx^\alpha)\otimes\left({}_0D^{\alpha}_{x}\right)\left[t\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]=\int_0^t\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes f_\alpha(tx^\alpha)\right]dt\,. \tag{16}$$

Proof Let $f_\alpha(x^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}$, then

$$f_\alpha(tx^\alpha)=\sum_{k=0}^{\infty}\frac{a_k}{k!}\left(t\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}. \tag{17}$$

Hence,

$$\left({}_0I^{\alpha}_{x}\right)\left[f_\alpha(tx^\alpha)\otimes\left({}_0D^{\alpha}_{x}\right)\left[t\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$
$$=\left({}_0I^{\alpha}_{x}\right)\left[\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\otimes\left({}_0D^{\alpha}_{x}\right)\left[t\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right]$$
$$=\sum_{k=0}^{\infty}\frac{a_k}{k!}t^{k+1}\left({}_0I^{\alpha}_{x}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\right]$$
$$=\sum_{k=0}^{\infty}\frac{a_k}{(k+1)!}t^{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}. \tag{18}$$

On the other hand,

$$\int_0^t\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes f_\alpha(tx^\alpha)\right]dt$$
$$=\int_0^t\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes k}\right]dt$$
$$=\int_0^t\left[\sum_{k=0}^{\infty}\frac{a_k}{k!}t^k\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}\right]dt$$
$$=\sum_{k=0}^{\infty}\frac{a_k}{(k+1)!}t^{k+1}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes(k+1)}$$
$$=\left({}_0I^{\alpha}_{x}\right)\left[f_\alpha(tx^\alpha)\otimes\left({}_0D^{\alpha}_{x}\right)\left[t\frac{1}{\Gamma(\alpha+1)}x^\alpha\right]\right].$$

That is, the desired result holds. Q.e.d.

Theorem 3.3: *Suppose that* $0<\alpha\leq 1$, *and* $(-1)^\alpha$ *exists, then the* α*-fractional Gaussian integral*

$$\left({}_{-\infty}I^{\alpha}_{+\infty}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]=\sqrt{\pi}\,. \tag{19}$$

Proof By Lemmas 3.1 and 3.2,

$$\left[\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]\right]^2$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]\cdot\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes 2}\right)\right]$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\otimes E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes 2}\right)\right]$$

$$=\left({}_0I_{+\infty}^{\alpha}\right)\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+\left(\frac{1}{\Gamma(\alpha+1)}y^\alpha\right)^{\otimes 2}\right]\right)\right]$$

$$\left(\text{let } \frac{1}{\Gamma(\alpha+1)}y^\alpha = t\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)$$

$$=\int_0^{+\infty}\left[\left({}_0I_{+\infty}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)\otimes E_\alpha\left(-(t^2+1)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]\right]dt$$

$$=\int_0^{+\infty}\left[-\frac{1}{2(t^2+1)}E_\alpha\left(-(t^2+1)\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\Big|_0^{+\infty}\right]dt$$

$$=\int_0^{+\infty}\left[\frac{1}{2(t^2+1)}\right]dt$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{1}{t^2+1}dt$$

$$=\frac{\pi}{4}. \tag{20}$$

Therefore,

$$\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]=\frac{\sqrt{\pi}}{2}. \tag{21}$$

Thus,

$$\left({}_{-\infty}I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]$$

$$=2\left({}_0I_{+\infty}^{\alpha}\right)\left[E_\alpha\left(-\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)\right]$$

$$=\sqrt{\pi}.$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we find the fractional Gaussian integral. A new multiplication of fractional analytic functions plays an important role in this paper. In fact, our result is a generalization of the result in ordinary calculus. In the future, we will expand our research fields to fractional differential equations and applied mathematics.

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